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Buckling of longitudinal reinforcement in RC columns

Flambagem de armaduras longitudinais em pilares de concreto armado

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Abstract

The main objective of the present work is to develop a formulation to analyze the buckling of longitudinal bars in reinforced concrete columns taking into account the tie spacing, the diameter and arrangement of the ties in the cross section and the longitudinal bar diameter. For this purpose an analytical method for the evaluation of the buckling load of longitudinal bars is described, as a function of the constraint imposed by the axial or flexural stiffness of the stirrups. The longitudinal bar is considered as a column deforming according to thin beam theory. The tie action is described either by a set of discrete elastic supports or by a continuous elastic foundation. The theoretical analysis considers the column with one or more deformation modes, with some degree of nonlinearity. As a result of this study, rational criteria for spacing and sizing of transversal reinforcement are derived, allowing the study of different alternatives using a simple design chart. © 2005 IBRACON. All rights reserved.

Keywords: Buckling; longitudinal and transverse reinforcement; columns; reinforced concrete.

Resumo

O presente trabalho apresenta um estudo sobre a flambagem das armaduras longitudinais em pilares de concreto armado submetidos a carregamento axial levando em conta o espaçamento entre estribos, o diâmetro e arranjo dos estribos na seção transversal e o diâmetro das armaduras longitudinais. Para este propósito um método analítico para a avaliação da flambagem da armadura longitudinal é proposto, considerando-se as barras longitudinais restringidas pela rigidez axial ou à flexão dos estribos. Admite-se que a armadura longitudinal funciona como uma coluna esbelta. Consideram-se duas formas de modelagem da atuação dos estribos: como apoios elásticos discretos e como base elástica contínua. O presente trabalho trata a coluna com um ou mais modos de deformação, incluindo certas não-linearidades. Como resultado deste estudo, apresenta-se uma proposta para dimensionamento racional dos estribos que permite estudar diferentes alternativas em um ábaco de utilização simples para projeto. © 2005 IBRACON. All rights reserved.

Palavras-chave: Flambagem; armaduras longitudinais e transversais; pilares; concreto armado.

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1 Introduction

The study of the instability of the longitudinal reinforcement in concrete reinforced columns has attracted some attention of researchers in recent years. However, most of the studies are restricted to the analysis of a buckling wave occurring between two consecutive stirrups, and consequently the interaction of the ties with the buckling of the longitudinal reinforcement is disregarded.

One of the first analytical studies about the instability of the longitudinal reinforcement in structures of reinforced concrete was made by Bresler & Gilbert [2]. They used criteria of elastic analysis to find relationships between the buckling of the reinforcement and parameters such as spacing and rigidity of the ties. A similar study was accomplished by Vallenas et al. [14] and Scribner [13], who considered that the maximum spacing of the ties for the calculation of the critical load of the column is such that the buckling wave length tends to be the same as the tie spacing, disregarding the overall buckling phenomenon.

The experimental study of reinforced concrete columns, with the purpose of visualizing the behavior of the longitudinal reinforcement has attracted the attention of several researchers in the last forty years. We mention the works of Pfister [8], Vallenas et al. [14], Kaar & Corley [5], Sheikh & Uzumeri [11], Scott et al. [12] and Moehle & Cavanagh [6], which consider the buckling of longitudinal reinforcement in heavily confined concrete columns involving the stirrups.

In face of the above state of affairs, a simple rational design methodology of the tie system appears to be necessary. A research has been developed by Buffoni [3], studying the buckling of longitudinal bars in reinforced concrete columns submitted to axial load, taking into account the tie spacing, the diameter and arrangement of the ties in the cross section and the longitudinal bar diameter.

For this purpose an analytical method is presented, allowing for the evaluation of the buckling load as a function of the constraint imposed by the axial or flexural stiffness of the ties. Two particular models were studied. In the first, the longitudinal bar is considered as a column and the stirrups as discrete supports. In the second case, the longitudinal bar as is taken a column on elastic foundation, where the elastic foundation is provided by the ties.

The case of splicing of the bars in the present formulation may be considered by taking the bar to be free in one of the extremities. The theoretical analysis considers the column with one or more deformation modes with some degree of nonlinearity. From the results of these analysis emerges a criterion for rational design, allowing for the choice of tie spacing, tie diameter and arrangement of the reinforcement in the cross section.

The deduction of the mathematical models that will be approached is not the objective of this article, which focus on the practical applicability of such models directly, through comparisons with experimental results of the literature. The hypotheses and formulations of each model are discussed in more detail in the thesis by Buffoni [3].

2 Formulation

The longitudinal bar is considered as a column where the ties can be represented schematically as intermediate elastic supports, whose stiffness K depends on the geometry of the arrangement and on the mechanical characteristics of the steel. The foundation stiffness is assumed constant. The model assumed for the determination of the critical load is shown in Figure 1, where L indicates the length of the bar, s is the tie spacing. The F_j are the forces corresponding to the elastic supports j, computed from where w_j is the displacement of the generic support.

$$F_j = Kw_j \Rightarrow w_j = \frac{F_j}{K} \tag{1}$$

In the development to follow, the usual hypotheses of Euler-Bernoulli for plane bending of thin beams are adopted. The column and the load are in a plan of symmetry and the cross section remains plane and perpendicular to the axis, before and after the deformations.

2.1 Strain energy and potential energy of the beam-column

The elastic stiffness matrices and the geometric stiffness matrices of the column are obtained starting from the strain energies and potential energy, respectively. All the mathematical formulation is accomplished starting from the column of Euler, simply supported and submitted to purely axial load. The deformation mode is chosen to satisfy the boundary conditions of the model presented in the Figure 1.

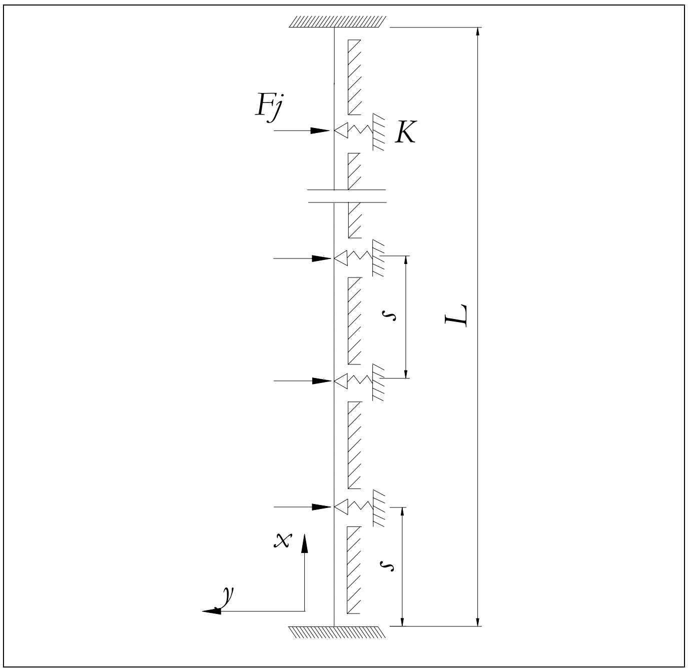

Figure 1 - Mathematical model of longitudinal and transversal reinforcement.

The strain energy is represented by the sum of the membrane strain energy, U_m, originating from the axial deformation of the bar, with the strain energy U_f, due to bending of the bar, leading to the expression below.

$$U = U_m + U_f = \frac{1}{2}\int_0^L EA_s\varepsilon_{xo}^2 dx + \int_0^L \frac{1}{2} EI\chi^2 dx \qquad (2)$$

In usual buckling problems, the strain due to bending is much more important than the axial strain and in the formulation of the problem the axial strain is disregarded. This hypothesis is adopted in the inextensional theory of beams presented in the works of Dym & Shames [4] and Bazant & Cedolin [1], where the bending strain energy of the beam is given by

$$\begin{aligned} U &\cong \int_0^L \frac{1}{2} EI\chi^2 dx \cong \int_0^L \frac{1}{2} EIw_{,xx}^{\;2}\left[1+\frac{1}{2}w_{,x}^{\;2}\right]^2 dx \\ &\cong \int_0^L \frac{1}{2} EI\left(w_{,xx}^{\;2} + w_{,xx}^{\;2}w_{,x}^{\;2} + \frac{1}{4}w_{,xx}^{\;2}w_{,x}^{\;4}\right)dx \end{aligned} \qquad (3)$$

In agreement with Dym & Shames [4], the approach shown in the expression (3) appears adequate to describe with accuracy the pos-critical equilibrium paths of the column, even for large transverse displacements.

In the adopted model, the stirrups are considered as linear springs with strain energy given by

$$U_i = \frac{1}{2}K\left[\sum_{i=1}^{i=n} w\left(x = is\right)\right]^2 \qquad (4)$$

where w is the value of the displacement in the points where stirrups exist and n is the number of stirrups involved in the model.

If we assume that the discrete elastic supports can be substituted by an elastic foundation with distributed stiffness $k=K/s$, the term corresponding to the strain energy of the ties in expression (4) can be evaluated as an integral in the following way:

$$\frac{1}{2}K\sum_{i=1}^{i=n} w\left(x = is\right)^2 \cong \frac{1}{2}\int_0^L kw\left(x\right)^2 dx \qquad (5)$$

The potential energy of the external loads is given by the product of the load, P, and the shortening in the extremity of the column, Δ, could be expressed as

$$V_p = P\Delta \Rightarrow V_p = -P\int_0^L\left(\frac{1}{2}w_{,x}^{\;2} + \frac{1}{8}w_{,x}^{\;4}\right)dx \qquad (6)$$

where P is the axial load and the negative sign appears due the loss of potential energy.

2.2 Non-dimensional variables

- **Discrete case**

In order to accomplish a parametric analysis, the following changes of variables will be made, using the following convenient parameters:

$$\xi = \frac{x}{L} \quad 0 \le \xi \le 1 \qquad w_d = \frac{w}{L} \qquad \Gamma = \frac{PL^2}{EI} \qquad \eta = \frac{KL^3}{EI} \qquad (7)$$

where ξ is the non-dimensional coordinate, w_d is the non-dimensional lateral displacement of the column, Γ is the non-dimensional axial load and η is the non-dimensional stiffness of the ties.

- **Column on elastic base**

The non-dimensional variables are the same ones considered in the expression (7), except for the stiffness parameter of the ties that is given by

$$\eta = \frac{kL^4}{EI} \qquad (8)$$

2.3 Deformation mode

The deformation mode of the column representing a bar is approximated by a function of the type:

$$w\left(x\right) = \sum_{j=1}^{n} A_j y_j\left(x\right) \qquad (9)$$

where n is the total number of degrees of freedom j, A_j are the modal amplitudes and the function are the modal functions. The functions $y_j(x)$ should satisfy the essential and when possible the natural boundary conditions of the column, with proper displacement restraints and moment zero in the extremities of the column, in order that the numerical solution converges better approximates the solution of the original problem.

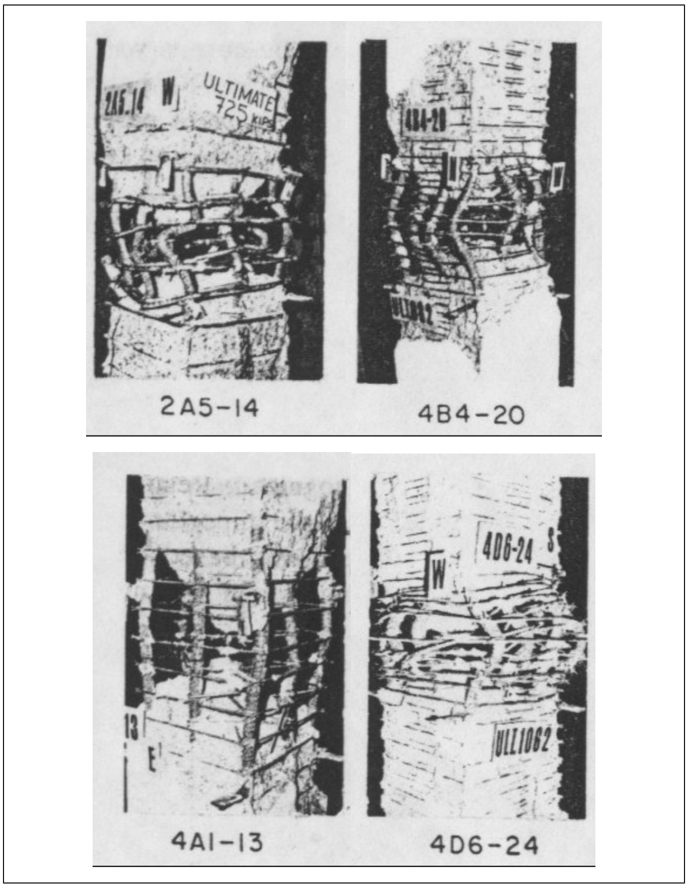

Figure 2 - Appearance of columns after tests (apud Sheikh & Usumeri [11]).

Experimental observations show that the buckling of the longitudinal bar can occur in a sinusoidal shape, that can involve several ties. The following function is taken to describe the deformation mode of the column:

$$w(x) = \sum_{m=1}^{\infty} A_m \left(-\frac{m\pi x}{L} + \frac{m\pi x^2}{L^2}\left(2+(-1)^m\right) - \frac{m\pi x^3}{L^3}\left(1+(-1)^m\right) + \sin\left(\frac{m\pi x}{L}\right)\right) \quad (10)$$

The above deformation mode combines functions that describe a symmetrical behavior with nonsymmetrical functions. The usual consideration of deformations just of the symmetrical type may lead to certain deviation, because it is verified in many cases that the buckling can involve non-symmetrical modes as displayed in the Figure 2 taken from the work of Sheikh & Uzumeri [11]. In non-dimensional form the deformation mode becomes:

$$w_d(\xi) = \sum_{m=1}^{j} a_m \left(-m\pi\xi + m\pi\xi^2\left(2+(-1)^m\right) - m\pi\xi^3\left(1+(-1)^m\right) + \sin(m\pi\xi)\right) \quad (11)$$

2.4 Integration along the bar and determination of elastic stiffness matrices (K_f) and geometric stiffness matrices (K_g)

The elastic stiffness matrices indicated by K_f are obtained from the strain energy expressed in Eq. (3); the stiffness matrices from the lateral supports, K_m, are obtained from the corresponding strain energy, Eqs. (4) and (5), and the geometric stiffness matrices, K_g are obtained from the energy of the axial load expressed in Eq. (6). In all cases we consider the field of displacements w_i and w_j for the given boundary conditions.

- **From strain energy**

$$K_{f_{i,j}} = \int_0^1 \left(w_{d,\xi\xi i} w_{d,\xi\xi j} + w_{d,\xi\xi i} w_{d,\xi\xi j} w_{d,\xi i} w_{d,\xi j} + \frac{1}{2} w_{d,\xi\xi i} w_{d,\xi\xi j} w_{d,\xi i} w_{d,\xi j} w_{d,\xi i} w_{d,\xi j} \right) d\xi \quad (12)$$

- **From strain energy of the lateral supports**

- *Discrete elastic supports*

$$K_{mi,j} = \eta \sum_{\xi=0}^{\xi=1} w_{di} w_{dj} \quad (13)$$

- *Continuous elastic foundation*

$$K_{mi,j} = \eta \int_0^1 w_{di} w_{dj} d\xi \quad (14)$$

- **From strain energy of the axial load**

$$K_{gi,j} = \int_0^1 \left(w_{d,\xi i} w_{d,\xi j} + \frac{1}{4} w_{d,\xi i} w_{d,\xi j} w_{d,\xi i} w_{d,\xi j} \right) d\xi \quad (15)$$

2.5 Solution of the eigenproblem

Taking into account the quadratic portion in the expressions (12) to (15), a linear eigenproblem is obtained, given by the following expression:

$$(K_f + K_m - \Gamma K_g)y = 0 \quad (16)$$

For a one-degree-of-freedom system the solution of equation (16) leads to the values of the critical load as follows:

$$\Gamma = K_g^{-1}\left(K_f + K_m\right) \quad (17)$$

For a multi-degree-of-freedom system an eigenvalue problem is solved. The computational program for the calculation of the eigenvalues comes in Buffoni [3].

2.5.1. Critical load parameters

Starting from the solution of the eigenproblem, is possible to find the parameters of critical load for the discrete and continuous cases.

In the discrete case, is considered that the length of the bar involved in the buckling varies from one to sixteen tie spacing. The deformation mode is introduced, with one or more degrees of freedom as described in (11), in the expressions for the evaluation of the stiffness matrices. Hence the eigenproblem is solved for the non-dimensional critical load values. The values for these cases are in the work of Buffoni [3].

In the continuous case, considering the longitudinal bar as a column on elastic foundation, where the elastic substract is composed by the stirrups, the deformation mode described in (11) with an or more degrees of freedom are introduced in the quadratic portion of the expressions (12), (14) and (15). The stiffness matrices then obtained and the eigenproblem is solved leading to the critical load. Other details can be found in Buffoni [3]. The expression (18) presents the critical load parameter for just one term in the modal expansion.

$$\Gamma = \frac{15\pi^6 - 120\pi^4 + 15\pi^2\eta + \pi^4\eta - 240\eta}{5\left(5\pi^2 - 48\right)\pi^2} \quad (18)$$

2.6 Consideration of the splicing of reinforcement

To take into account splicing of the bars in the present formulation, where there may be loss of continuity of displacements, it is suggested a crude model where the reinforcement is fixed in one of the extremities and free in the other as the model presented in the Figure 3. In that way, the steps accomplished for the column supported in the extremities can be repeated with the consideration of splicing, allowing for an estimate of the buckling behavior of the longitudinal bars in such a condition.

2.7 Calculation of the stiffness non-dimensional parameter η

The value of the stiffness non-dimensional parameter in the model depends on the stiffness of the ties, K, the modulus of elasticity of the longitudinal bar, E, the moment of inertia of the longitudinal bar, I, and the tie spacing s.

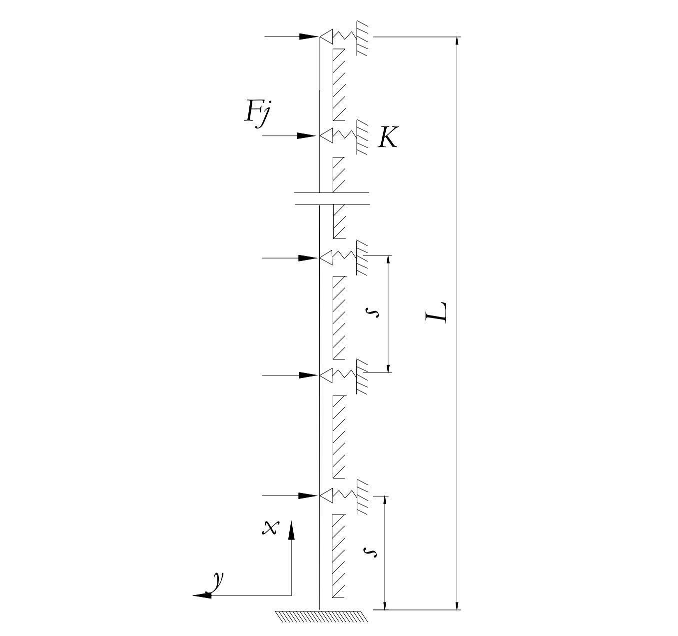

Figure 3 - Model of longitudinal splicing.

2.8 Calculation of the stiffness of the ties, *K*

The value of *K* is a function of the geometry, arrangement and mechanical characteristics of the tie. Several arrangements have been considered in this formulation. The Figure 4 presents some cases. A purely axial load with complete symmetry is assumed. The value of *K* is calculated with the simplified models of the Figure 5.

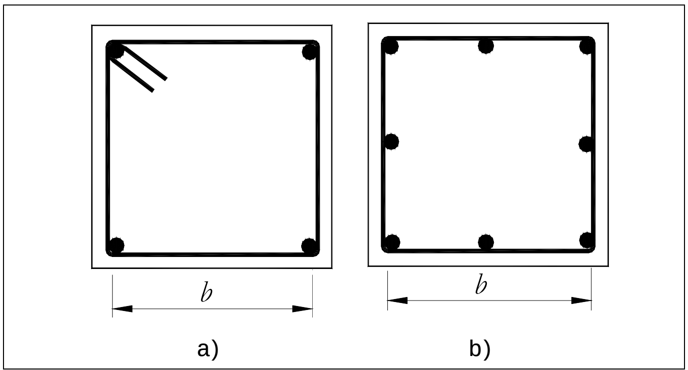

Figure 4 - Transverse reinforcement arrangements.

In the model of the Figure 5.a the longitudinal bar is constrained by the flexural stiffness of the tie. In the Illustrations 5.b and 5.c the longitudinal bar is constrained to a greater extent by the axial stiffness of the tie. The Figure 5 also shows the force exerted by the longitudinal bar on the stirrup in the direction buckling can happen.

For the model of the Figure 5.a, the longitudinal bar can be considered as imposing a concentrated load in the middle of the span of a fixed beam in the extremities and the expression for the stiffness of the tie is given by

$$K = \frac{192EI_t}{b^3} \qquad (19)$$

For the model of the Figure 5.b the expression for the stiffness of the tie is

$$K = \frac{EA_t}{b} \qquad (20)$$

where *E* is the modulus of elasticity of the longitudinal bar, I_t is the moment of inertia of the tie and A_t is the area of the cross section of the tie.

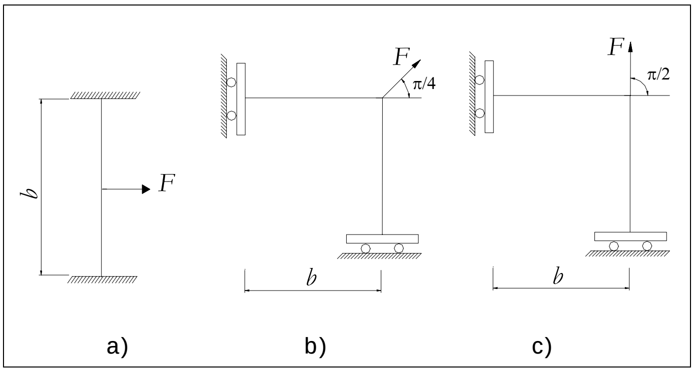

Figure 5 - Simplified models for stiffness calculation.

It should be noticed that when the arrangement of the bars in the cross section is similar to the one in Figure 4.b, the bars located in the center of a leg of stirrups buckle first. We remark further that the stirrups can offer different contributions for the buckling strength of the longitudinal bars. The longitudinal bars located in the corners of the stirrups are restricted by extensional stiffness of the stirrups and the bars located along the legs of the stirrups are restricted mostly by the bending stiffness of the stirrups.

The longitudinal modulus of elasticity longitudinal considered herein is the usual one; a reduced modulus of elasticity can be used if convenient.

3 Curves for the calculation of spacing and diameter of ties

Figure 6 presents the curve that relates the critical load parameter of the column, and the stirrups stiffness parameter, with the contributions of several modes for the buckling load. It is noticed that this graph presents a curve when the bar is fixed in the extremities and another curve when one of the extremities is free to simulate the presence of splicing of the bars.

It is verified that starting from a high stiffness level, the buckling load increases almost in linear proportion with the increase of the stiffness of the stirrups. Based on this graph a method will be presented for the calculation of the spacing and diameter of the stirrups of the reinforced concrete columns.

The curve of Figure 6 is valid for any type of arrangement of the reinforcement. According to the objective of the design, it is enough to introduce the respective values of Γ or η for each case under consideration.

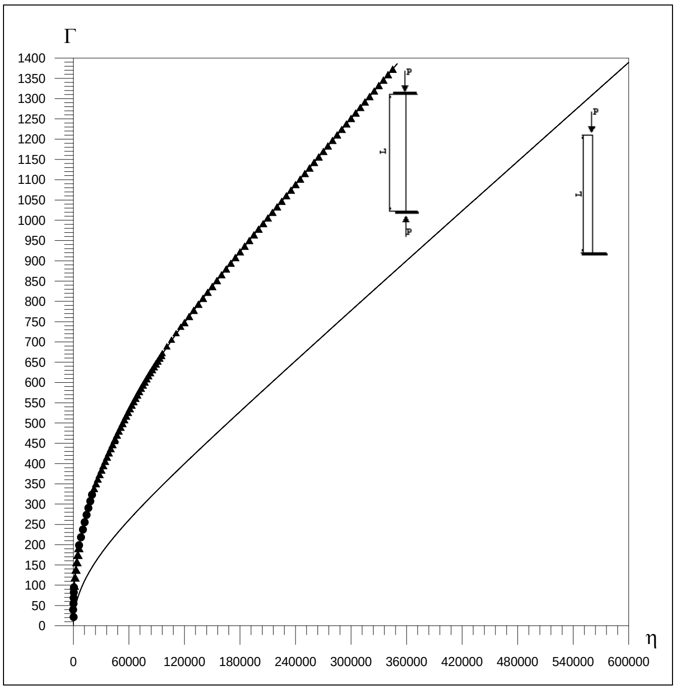

Figure 6 - Load parameter *vs.* stiffness parameter.

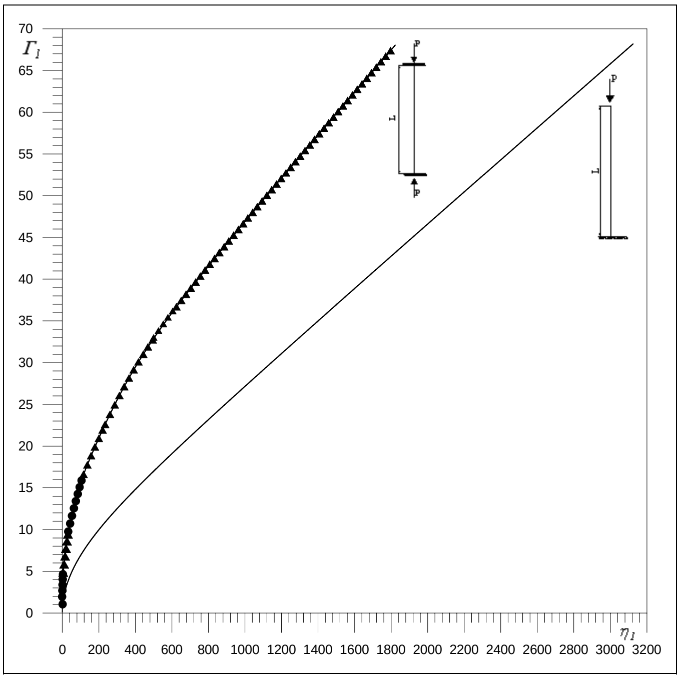

Figure 7 - Load parameter vs. stiffness parameter.

For example, for the cross section of the Figure 4.b, introducing the value of the stiffness of the ties, *K*, from equation (19), the value of the non-dimensional stiffness parameter, η, expressed in (8), is obtained by

$$\eta = \frac{192\phi_t^4 L^4}{b^3 s \phi_l^4} \qquad (21)$$

On the other hand, introducing the moment of inertia of the longitudinal bar in the expression for the load parameter Γ :

$$\Gamma = \frac{64PL^2}{E\pi\phi_l^4} \qquad (22)$$

Hence it is possible to adjust the values of the Γ and η of the expressions (21) and (22) or to create new parameters with the purpose of facilitating the calculations of the examples that will be presented. The non-dimensional variables Γ and η were modified in the following way:

$$\eta_1 = \frac{\eta}{192} = \frac{\phi_t^4 L^4}{b^3 s \phi_l^4} \qquad (23)$$

$$\Gamma_1 = \frac{\Gamma\pi}{64} = \frac{PL^2}{E\phi_l^4} \qquad (24)$$

Therefore, the ordinate and abscissa of the graphs of the Figure 6 are altered multiplying the same ones for the factors, $\frac{\pi}{64}$ and $\frac{1}{192}$, respectively. In that way, it is obtained the graph of the Figure 7. Depending on the arrangement of the stirrups in the cross section, a different value is taken for the stiffness *K* and for the non-dimensional stiffness parameter of the lateral supports η .

3.1 Considerations on the buckling load for rational design

The buckling load should always be larger than the compressive yield load for a factor $\gamma > 1$, thereby assuring the full capacity $P_y = f_y A_s$ in the initial design, as it is usually done. In the usual naïve design, one might attempt to optimize the design taking $\gamma = 1$, which is little advisable from the point of view of safety in the post-critical regime. The present work uses the value $\gamma = 1,2$.

4 Design sequence with the use of the Γ *x* η curves

A possible sequence of project with the use of the graphs Γ *x* η can proceed as follows:

a) the value of *b* comes from the geometry of the piece;

b) the diameter of the longitudinal bar, ϕ_l is decided;

c) a buckling load is looked for, $P_{cr} = \gamma \cdot P_y$ and Γ_1 is obtained;

d) with the value of Γ_1 one enters in the ordinate of the graph Γ_1 *x* η_1 and obtains the required η_1 in the abscissa. Since the values of *b* and ϕ_l are already available, the remaining design variables will be the tie spacing *s* and the tie diameter ϕ_t , which are expected to be compatible with existing project norms. In case the spacing results too small, or diameter to big, is necessary to reduce *b* or to use supplemental stirrups to stiffen the section.

4.1 Calculation of the spacing and sizing of transversal reinforcement for the columns described in the work of Queiroga & Giongo [10]

Queiroga & Giongo [10] studied columns of square section with the arrangement of the reinforcement in the cross section shown in the Figure 8. The columns P1, P4 and P6 were selected for the accomplishment of the numeric tests herein.

The calculated values for the tie diameter and tie spacing using the proposed criterion in the item 4 it is shown in Table 1. The complete procedure of calculation is described in Buffoni [3]. The steps for the column P1 are presented next.

The properties of reinforcement of the column P1 are given:

$f_y = 502 \; N/mm^2$ $\phi_l = 12,5 \; mm$

$L = 1200 \; mm$ $b = 139,9 \; mm$

$E = 210000 \; N/mm^2$ $A_s = 125 \; mm^2$

$s = 150 \; mm$ $P_y = f_y A_s = 5,46 \times 10^4 \; N$

The objective here is to calculate the tie diameter and tie spacing for $\gamma = 1,2$ considering the reinforcement without splicing.

A required buckling load is taken with $P_{cr} = \gamma P_y$ resulting in a value for Γ_1.

$$P_{cr} = \gamma P_y = 65,48 \; kN \Rightarrow$$

$$\Gamma_1 = \frac{P_{cr} L^2}{E \phi_l^4} \Rightarrow \Gamma_1 = 18,39 \Rightarrow \eta_1 = 149,10 \tag{25}$$

The value of η_1 found in the expression (25) is obtained by entering the value of Γ_1 in the ordinate of the graph presented in the Figure 7 and extracting η_1 from the corresponding abscissa for Γ_1. Starting from the expression (23) one finds

$$\frac{\phi_t^4}{s} = \frac{\eta_1 b^3 \phi_l^4}{L^4} = 4,81 \tag{26}$$

Table 1 presents the values obtained for the columns P4 and P6 that were calculated in the same way that the column P1. In the line corresponding to the column P1, the value found for the diameter being considered a spacing $s = 150 \; mm$ is approximately $\phi_t = 5,20 \; mm$.

It is noticed that these values are found starting from the buckling mode that might involve several stirrups in a limiting state. The Table 1 presents the commercial values for the tie diameter.

It is also verified that when the tie spacing diminishes the value of the tie diameter could be smaller. The Table 1.b presents the case where one of the extremities of the bar is free. In this case, the values found for the diameter of the stirrups are higher, because a certain value of the load parameter will require higher values of stiffness; consequently, higher values for the diameter of the stirrups are necessary.

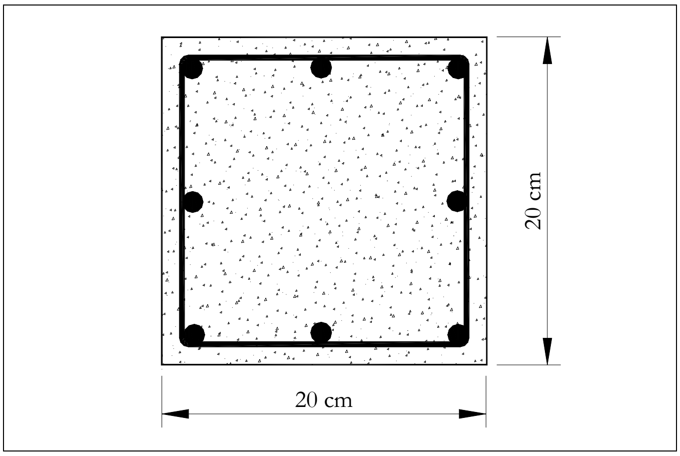

Figure 8 - Cross section and arrangement of reinforcement in the columns tested by Queiroga [9]

Table 1 - Calculation of the tie diameter and tie spacing for the columns of Queiroga [9] starting from the proposed formulation.

a) Longitudinal reinforcement fixed in the extremities

Queiroga [9]		Bar fixed in the extremities				
Column	Transversal Reinforcement	Γ_1	η_1	Transversal Reinforcement: s mm	ϕ_t mm	ϕ_{tcom} mm
P1	$\phi 6,3c/15$	18,4	149,1	150	5,20	6,3
P4	$\phi 6,3c/10$	18,4	149,1	100	4,68	5
P6	$\phi 6,3c/5$	18,4	149,1	50	3,94	5
Note: for tie diameter, ϕ_t=6,3 mm, maximum spacing is 328 mm.						

b) Free bar in one of the extremities

Queiroga [9]			Free bar in one of the Extremities		
Column	Γ1	η1	Transversal Reinforcement: s mm	ϕt mm	ϕtcom mm
P1	18,4	566,3	150	7,23	8
P4	18,4	566,3	100	6,54	8
P6	18,4	566,3	50	5,50	6,3
Note: for tie diameter ϕ_t=6,3mm , maximum spacing is 83mm.					

In agreement with NBR 6118/2003 the value of the diameter of the stirrups should be:

$$\phi_t \geq \begin{cases} 5mm \\ \phi_l / 4 \end{cases} \tag{27}$$

The spacing should be such that in a collapse state the buckling would happen among two consecutive stirrups. NBR 6118/2003 presents the following limits:

$$s \le \begin{cases} 20\ cm \\ b, menor\ dimensão\ da\ seção \\ 24\phi_l\ para\ CA-25, 12\phi_l\ para\ CA-50 \end{cases} \qquad (28)$$

The first two limits of the expression (28) correspond to dispositions merely constructive and the last two are found starting from the condition, $f_{cr} = f_y$; in other words, the tie spacing should allow the buckling critical stress to be the same as the yield stress. For the bars of steel *CA-25*, starting from the formulas of the materials strength (diagram linear stress-strain), one obtains

$$f_{cr} = \frac{P_{cr}}{A_s} = \frac{EI\pi^2}{s^2 A_s} = \frac{E\pi^2}{\lambda_{cr}^2} \qquad (29)$$

where $\lambda_{cr}^2 = s^2/r^2$, s is the tie spacing e r is the radius of gyration given for

$$r = \sqrt{\frac{I}{A_s}} \qquad (30)$$

The value of the critical stress is

$$f_{cr} = \frac{250}{1,15} = 217,4\ ^{N}\!/_{mm^2} \qquad (31)$$

This leads to the following values for the tie spacing, whose average values are prescribed in the code:

$$\lambda_{cr} = \sqrt{\frac{\pi^2 210000}{217,40}} \Longrightarrow \lambda_{cr} r = s \Rightarrow$$
$$\Rightarrow s = 97,64\frac{\phi_l}{4} = 24,41\phi_l \qquad (32)$$

NBR 6118/2003 includes the value $24\phi_l$ for the steel *CA-25*. In the case of the steel *CA 50*, the buckling critical tension is given by the formula of Tetmajer.

$$f_{cr} = 480\left(1 - 0,0035\lambda_{cr}\right) \qquad (33)$$

Making $f_{cr} = 400\ ^{N}\!/_{mm^2}$ it is arrived

$$\lambda_{cr} \cong 47,6 \Rightarrow s = \lambda_{cr} r = 47,6\frac{\phi_l}{4} = 12\phi_l \qquad (34)$$

It allows adopting $\phi_t < {}^{\phi_l}\!/_4$ since the tie spacing also respects the limitation

$$90000\left(\phi_t^2/\phi_l\right)\frac{1}{f_{yk}} \qquad (35)$$

where ϕ_l e ϕ_t are respectively the diameters of the longitudinal bar and of the stirrups, f_{yk} it is the yield stress of the longitudinal bar, in MPa. Those criteria suppose that both the tie and longitudinal bar are constituted by the same type of steel.

The expression (35) guarantees, in the case of adoption of $\phi_t < \phi_l/4$, the existence of same percentage of the volumetric ratio of stirrups that we would have with $\phi_t = \phi_l/4$ and spacing $24\phi_l$ (*CA 25*) and $12\phi_l$ (*CA 50*).

By considering the tie spacing $s = 24\phi_l$ (*CA 25*) the volumetric percentage ρ_v of the ties (length *2p*) for volume of the column given:

$$\rho_v = \frac{\frac{\pi}{4}\left(\frac{\phi_l}{4}\right)^2 2p}{24\phi_l A_c} = \frac{\pi\phi_l 2p}{1536 A_c} \qquad (36)$$

In case it is adopted $\phi_t < \phi_l/4$, for the maintenance of the same value of ρ_v , the tie spacing s' is given by

$$\frac{\pi\phi_t^2 2p}{4s' A_c} = \frac{\pi\phi_l 2p}{1536 A_c}, \quad s' = 384\frac{\phi_t^2}{\phi_l} \qquad (37)$$

Following an identical reasoning, we would arrive to the limit $192\frac{\phi_t^2}{\phi_l}$ for the steel *CA 50*. It is noticed that the appearance of the value 384 in the expression (37) for the steel CA 25 corresponds to the term $90000/f_{yk}$ in the expression (35), in other words,

$$90000/250 \approx 360\ \ para\ CA\ 25$$
$$90000/500 \approx 180\ \ para\ CA\ 50 \qquad (38)$$

In agreement with the several design codes, these values consider the ultimate limit states, that the buckling of the longitudinal bar would happen mostly between stirrups, because is considered that the buckling length is the tie spacing.

In that way, in agreement with NBR 6118/2003 the appropriate values for the spacing and diameter of the stirrups for the columns studied by Queiroga [9] are:

$$\begin{cases} s \le 12\phi_l \Rightarrow s = 150\ mm \\ \phi_t \ge 5 \quad \Rightarrow \phi_t = 5\ mm \end{cases} \qquad (39)$$

4.2 Applications to sections of great dimensions

The proposed criterion is applied for reinforced concrete columns with rectangular cross section 25cm x 110cm, span free from 350cm, with compressive strength of concrete 20MPa and of the steel of 500MPa. The longitudinal bars consist of 22 bars of 16mm, and the cover is of 3cm. The column is named P1 and some cases of variations in the arrangements of the reinforcement in the cross section are presented.

- **Case 1**

Consider the arrangement of the reinforcement in the cross section presented in Figure 9.

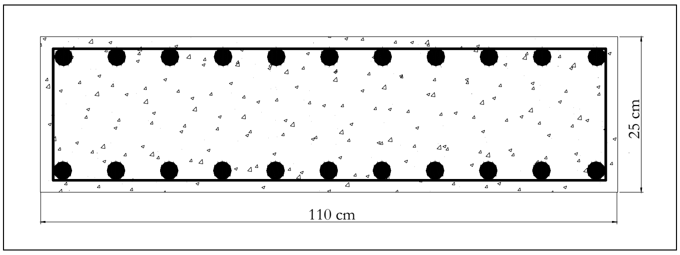

Figure 9 - Load parameter *vs.* stiffness parameter.

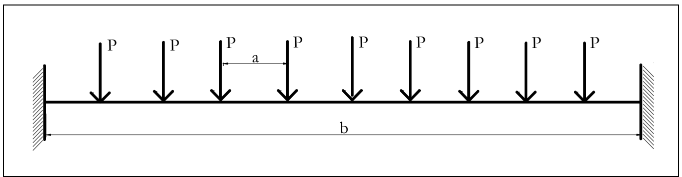

Figure 10 - Simplified model for stiffness calculation in case 1.

For the calculation of the stiffness *K* of the ties the model of the Figure 10 is adopted, where the leg of the stirrup is considered as a beam fixed in the ends. The flexibility of the stirrup associated to each bar can be obtained being applied a concentrated unitary load in each central point of the bar. It is noticed that this corresponds to admit that the onset of buckling occurs in the least restrained bar. The critical point is evaluated for the bar that counts with the smallest stiffness of the stirrup. For the model of the Figure 10 this happens in the center of the beam, and in this case the stiffness becomes

$$K = \frac{38,4EI_t}{b^3} \quad (40)$$

where the span free from bending is $b = 1100 - 2(30+5) - 16 = 1014\ mm$, assuming the diameter of the stirrup initially to be 5mm. Considering the stirrups as elastic foundation and substituting *k* = *K/s* in the parameter η we arrive at

$$\eta = \frac{kL^4}{EI} = \frac{38,4EI_tL^4}{b^3sEI} = \frac{38,4\phi_t^4L^4}{b^3s\phi_l^4} \Rightarrow$$
$$\frac{\phi_t^4}{s} = \frac{\eta b^3\phi_l^4}{38,4L^4} = \frac{192\eta_1 b^3\phi_l^4}{38,4L^4} \quad (41)$$

The objective is to calculate the tie diameter and tie spacing being considered the longitudinal bars without splicing. In that way, a buckling load is computed, $P_{cr} = \gamma P_y$, leading to Γ_1 and η_1.

$$P_{cr} = \gamma P_y = 104,4\ kN \Rightarrow \Gamma_1 = \frac{P_{cr}L^2}{E\phi_l^4} \Rightarrow$$
$$\Gamma_1 = 92,93 \Rightarrow \eta_1 = 2802,08 \quad (42)$$

Starting from the expression (23) it is had:

$$\frac{\phi_t^4}{s} = 6,38 \times 10^3 \quad (43)$$

Some values were stipulated for the tie spacing in agreement with the limits imposed by NBR 6118/2003. The required diameters which prevent buckling of the longitudinal bar are shown in the Table 2. The values found for the tie diameter were high, because this model is quite flexible.

Table 2 - Results for case 1.

S (mm)	190	150	50
ϕ_t (mm)	33,18	31,28	23,77

- **Case 2**

The considered model is shown in the Figure 11 where there is a supplemental stirrup (admitted as rigid extensionally) in the middle of a stirrup leg. The model simplified for the calculation of the stiffness is in Figure 12. The located loads in the distance 2a or 3a of the left support in the Figure 12 contribute with to smallest tie stiffness, and in this case the stiffness becomes

$$K = \frac{250EI_t}{3b^3} \quad (44)$$

where the bending free span is $b = 507\ mm$. Substituting the expression (44) in the value of η one obtains

$$\eta = \frac{kL^4}{EI} = \frac{83,33EI_tL^4}{b^3sEI} = \frac{83,33\phi_t^4L^4}{b^3s\phi_l^4} \quad (45)$$

and from expression (45)

$$\frac{\phi_t^4}{s} = \frac{\eta b^3\phi_l^4}{83,33L^4} = \frac{192\eta_1 b^3\phi_l^4}{83,33L^4} = 367,46 \quad (46)$$

Starting from the expression (46) the values obtained are presented in Table 3. In relation to the case 1, this model is more rigid, however still quite flexible. In agreement with the results presented in the Table 3, the values found for the tie diameter are still high. Thus an arrangement is looked for that is rigid enough to lead to reasonable values for the tie diameter and tie spacing.

Table 3 - Results for case 2.

s (mm)	190	150	50
ϕ_t (mm)	16,26	15,32	11,64

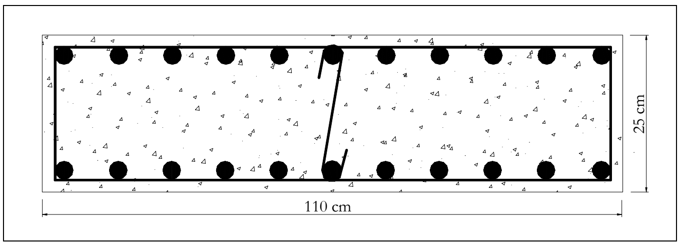

Figure 11 - Case 2: Arrangement of reinforcement in cross section of column P1.

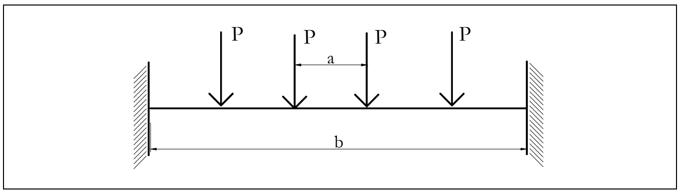

Figure 12 - Simplified model for stiffness calculation for case 2.

- **Case 3**

Figure 13 presents the arrangement of the reinforcement in the cross section. For the calculation of the tie stiffness *K*, the model adopted is presented in the Figure 14. The expression of the stiffness for the bar less restrained. i.e. more distant from the fixed support, is

$$K = \frac{6b^3EI_t}{a^3\left(21b^3 + 192a^2b - 128a^3 - 108b^2a\right)} \qquad (47)$$

where $b = 507\ mm$ and $a = 46\ mm$. The expression for the calculation of the design of the stirrups is shown in the expression (48), which is obtained starting from the expressions of η and K presented in (8) and (47), respectively.

$$\frac{\phi_t^4}{s} = \frac{32\eta_1\phi_l^4\left[a^3\left(21b^3 + 192ba^2 - 128a^3 - 108b^2a\right)\right]}{b^3L^4} \qquad (48)$$

It is noticed from Table 4 that the values found for the diameter, despite still high, are much smaller than those for case 2.

For all cases considered until the present, we considered $\gamma=1,2$ for the calculation of the buckling load. As the usual design admits the value of $\gamma = 1,0$, the calculations were repeated with such value and the results found is shown in the Table 4.

Table 4 - Results for case 3.

s (mm)	ϕ_t (mm)	
	$\gamma = 1,0$	$\gamma = 1,2$
190	9,21	9,79
150	8,68	9,22
50	6,59	7,01

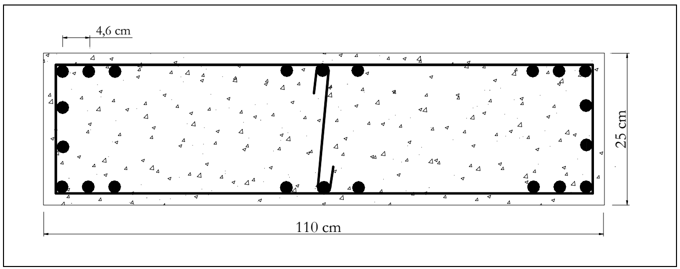

Figure 13 - Case 3: Arrangement of reinforcement in cross section of column P1.

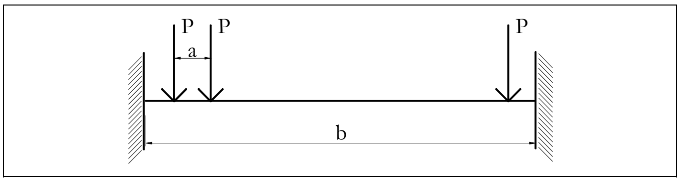

Figure 14 - Simplified model for calculation of stiffness in case 3.

- **Case 4**

The arrangement of the bars in the cross section is shown in the Figure 15. The models simplified for the calculation of the tie stiffness *K* are presented in Figure 16. Figure 16.a represents the first or last part of the cross section and Figure 16.b represents a region among supplemental stirrups. Is verified that the largest flexibility is for the bar of the center, and the value of the stiffness *K* is

$$K = \frac{96EI_t}{b^3} \qquad (49)$$

where $b = 184\ mm$. The expression for the calculation of the design of the stirrups obtained starting from the expressions of η and *K* presented in the expressions (8) and (49), respectively, is given by

$$\frac{\phi_t^4}{s} = \frac{2\eta_1\phi_l^4b^3}{L^4} \qquad (50)$$

The results are in the Tables 5 and 6.

Table 5 – Results for case 4.

s (mm)	ϕ_t (mm)	
	$\gamma = 1,0$	$\gamma = 1,2$
190	6,90	7,34
150	6,50	6,92
50	4,94	5,25

Table 6 – Results for case 4.

ϕ_t (mm)	s (mm)	
	$\gamma = 1,0$	$\gamma = 1,2$
5	53	41
6,3	132	103

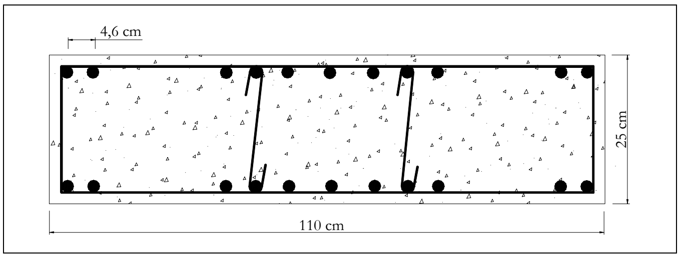

Figure 15 - Case 4: Arrangement of reinforcement in cross section of column P1.

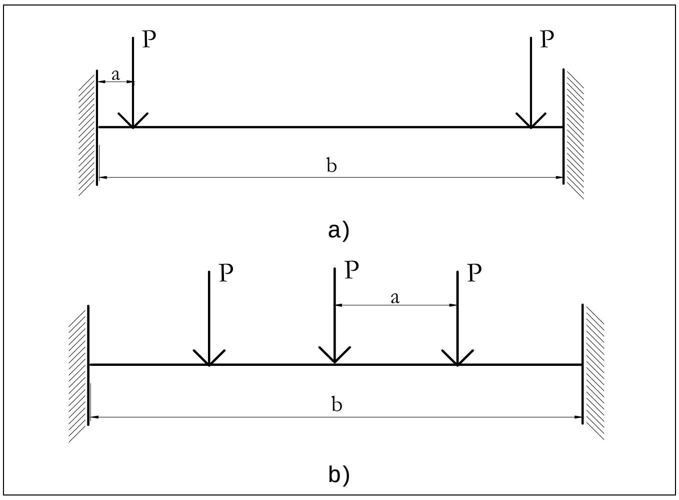

Figure 16 - Simplified model for stiffness calculation in case 4.

- **Case 5**

The model of Figure 17 turned out to be the most favorable and economical in terms of design of the stirrups. The simplified model used for the calculation of K is shown in Figure 18, from which results the value of $b = 338\ mm$, with K given by

$$K = \frac{6bEI_t}{a^3\left(2b-3a\right)} \tag{51}$$

The expression for the design of the stirrups, from the expressions of η and K presented in (8) and (51), respectively, it is given by

$$\frac{\phi_t^4}{s} = \frac{32\eta_1\phi_l^4 a^3\left(2b-3a\right)}{bL^4} \tag{52}$$

The results are in Table 7 for the values of $\gamma = 1,0$ and $\gamma = 1,2$. It is verified through the calculations that if one defines the tie diameter as $\phi_t = 6,3\ mm$, the tie spacing is larger than the upper limit registered in NBR 6118/2003. Starting from this model a study was undertaken to verify the possibility to execute it with double spacing.

The average stiffness takes the average of the values of the stiffness K with supplemental stirrups being used with the simplified model of the Figure 18, according to expression (51). The stiffness of the model of the Figure 17, without supplemental stirrups, is presented in the simplified model of the Figure 19.

The value of the stiffness, K, for the model of the Figure 19, corresponding to one of the closest loads of the center is given for

$$K = \frac{54bEI_t}{\left(b^4 + 3b^3a - 9a^2b^2 + 18a^3b - 81a^4\right)} \tag{53}$$

The value of $b = 1014\ mm$ and the results being considered the medium stiffness of the elastic base for the consideration of double spacing, with the values of $\gamma = 1,0$ and $\gamma = 1,2$ presented in Table 8.

Table 7 - Results for case 5.

Tie diameters

s (mm)	ϕ_t (mm)	
	γ=1,0	γ=1,2
190	5,48	5,83
150	5,16	5,49
50	3,92	4,17

Tie spacing

ϕ_t (mm)	s (mm)	
	γ=1,0	γ=1,2
5	131	103
6,3	331	259

Table 8 - Results for case 5, with alternate spacing.

a) Tie diameters

s (mm)	ϕ_t (mm)	
	γ=1,0	γ=1,2
190	6,51	6,93
150	6,14	6,53
50	4,96	4,96

b) Tie spacing

ϕ_t (mm)	s (mm)	
	γ=1,0	γ=1,2
5	66	51
6,3	166	129

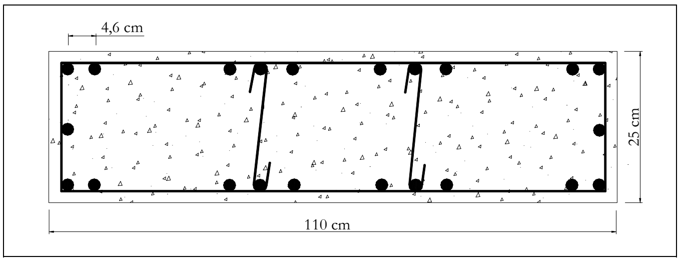

Figure 17 - Case 5: Arrangement of reinforcement in cross section of column P1.

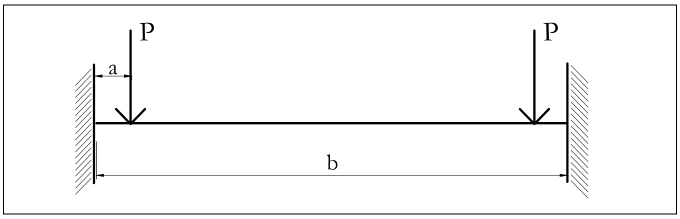

Figure 18 - Simplified model for calculation of stiffness in case 5.

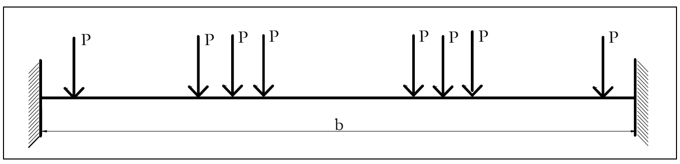

Figure 19 - Simplified model for calculation of stiffness in case 5, no supplementary stirrups.

5 Conclusions

The present work studied the buckling of the longitudinal reinforcement in reinforced concrete columns submitted to axial load, taking into account the tie spacing, the diameter and arrangement of the stirrups in the cross, section and the diameter of the longitudinal bars.

Graphs were presented that relate the critical load parameter of the longitudinal bar with the tie stiffness parameter. It is found that the value of the load parameter, for high values of the tie stiffness parameter, varies almost linearly. This suggests the adoption of simplified expressions for design rules. However, the procedure adopted in this work seems sufficiently simple for routine use.

Starting from the graphs that relate the load parameter of the longitudinal bar with the tie stiffness parameter, a sequence is proposed for sizing and positioning stirrups, characterizing a rational design of the stirrups in reinforced concrete columns.

The examples show that the result of such a procedure can become compatible with the existent results in current design codes.

The values obtained in the calculations for tie spacing and tie diameter consider general buckling that may involve several stirrups.

It has been shown, through examples, that the procedure adopted in this work allows for the necessary stiffness to be reached (that is, critical load above the compressive limit load limit of the longitudinal bar), with use of several design variables: tie spacing, tie diameter, diameter of the longitudinal bar, use of supplemental stirrups (in each section or alternately), and relocation of the bars of the longitudinal bar. The last two variables correspond to variations in the free span of the supporting stirrups. Several transversal reinforcement arrangements were discussed that supply larger stiffness and confinement. Such arrangements are more usual in projects looking for larger ductility, such as under seismic actions.

The Brazilian Code NBR 6118/2003 recommendations, as well as others, are intended to assure that buckling does not occur between two consecutive stirrups. However, the present study suggests that such restriction is over-conservative and may lead to unnecessarily high values of the load parameter Γ. In some instances, the approach used herein may require a denser reinforcement. Hence, a rational procedure such as the one proposed here may prove safer and more economical.

The case study presented in Figure 17 showed that the concentration of the longitudinal bars close to the corners causes an increase in the tie stiffness parameter, since there is a reduction of the bending span of the stirrups. Once more, the number of supplemental stirrups can be reduced in a rationally justifiable manner.

An effective stiffness can be taken for the stirrup as the average among the calculated stiffness when are used alternating supplemental stirrups to ease the placement of concrete.

In summary, a rational procedure for the design of transverse reinforcement has been introduced, with use of considerations related to the buckling of the longitudinal bars. A larger reinforcement requirement may be compensated by better conditions of execution of concrete columns, due to the justified reduction of supplemental stirrups. These and other aspects related to the function of the transverse reinforcement should be considered in the future design codes.

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