APPLICATION OF NONLINEAR FRACTURE MECHANICS TO THE ASSESSMENT OF ROTATIONAL CAPACITY IN REINFORCED CONCRETE BEAMS

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Outline

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- Snap-back instability in structural engineering
- Cohesive Crack Model
- Overlapping Crack Model
- Numerical algorithm for RC beams in bending
- Size-scale effects on the rotational capacity
- Size-scale effects on the minimum flexural reinforcement
Ductility in structural design

The development of ductility is a key parameter for the design of RC beams in bending.

The rotational capacity is required in order to:

• provide structural robustness;
• give warning of incipient collapse by the development of large deformation prior to collapse;
• allow the bending moment redistribution in statically indeterminate structures;
• enable major distortions and energy dissipation during earthquakes;
• withstand impact and cyclic load.

Different nonlinear contributions are involved in the flexural mechanical behaviour of RC elements.

Traditional models, based on stress–strain constitutive laws for concrete and steel, do not capture the final softening branch, and define the ultimate condition by imposing limits to the materials deformations instead of by means of the drop of the resistant moment.
1. The rotational capacity of RC beams in bending has been investigated from the experimental point of view in the early 1960s (CEB Indeterminate Structures Commission).

The following empirical hyperbolic relationship between plastic rotation and relative neutral axis depth was proposed:

\[ \varphi_{pl} = \frac{0.004}{x/d} \]

Model Code 78


2. Analytical and experimental research coordinated by prof. Eligehausen

Two different collapse mechanisms (steel yielding and concrete crushing) and different steel ductility classes have been considered.

Eurocode 2

- When the plastic analysis is adopted in structural design, it has to verify that the admissible rotation in the plastic hinge is greater than the required rotation.

\[ 0.6h \]

Experimental results

- The ultimate rotation is a decreasing function of the structural size.


Modelling the crack tip process zone in quasi-brittle and ductile materials

(a) CRACK TIP PROCESS ZONE

(b) COHESIVE FORCES behind the fictitious crack tip

Development of Cohesive Zone Models

Dugdale (1960) crack-tip plastic zone (metals)
Barenblatt (1962) cohesive atomic forces (crystals)
Bilby, Cottrell & Swinden (1963) crack-tip plastic zone (metals)
Rice (1968) crack-tip plastic zone (metals)
Smith (1974) analysis of different cohesive laws (metals and concrete)

Hillerborg et al. (1976) Fictitious Crack Model, for the analysis of Concrete (computational)
Carpinteri (1984-1989) Cohesive Crack Model, for the analysis of snap-back instabilities (quasi-brittle mat’s)


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The problem of snap-back instability

Strain softening represents a violation of the Drucker’s Postulate. As a consequence, the following phenomena may occur:

- **Loss of stability in the controlled load condition** (*snap-through*);
- **Loss of stability in the controlled displacement condition** (*snap-back*).


Snap-back instabilities of the elastic equilibrium in thin shells


• Case 1: $a = 0$
  The load-deflection relation is linear: $\tilde{P} = \frac{4}{\lambda^2} \tilde{\delta}$, for $\tilde{P} \leq 2/3$ ($\sigma \leq \sigma_u$)

• Case 2: $a = h$
  The following equilibrium scheme can be considered:

![Equilibrium scheme diagram]

The load-deflection relation is hyperbolic ($s_c = G_f/\sigma_u h$; $\lambda = l/h$):

$$\tilde{P} = \frac{1}{6} \left( \frac{s_c \lambda^2 \tilde{\delta}}{\tilde{\epsilon}} \right)^2, \text{ for } \tilde{P} \leq 2/3 \quad (x \leq h)$$

Both equations have the same upper limit: $\tilde{P} \leq 2/3$.

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• By transforming the load bounds into deflection bounds:

$$\tilde{\delta} \leq \delta_1 = \frac{\lambda^3}{6} \quad (1) \quad \tilde{\delta} \geq \delta_2 = \frac{s_c \lambda^2}{2 \tilde{\epsilon}_u} \quad (2)$$

a stability criterion for elastic-softening beams is obtained:

![Stability criterion diagram]

• When the two domains are disjoint, the $P-\delta$ curve is regular;

• When they are partially overlapped, it is well-founded to suppose them connected by a curve with highly negative or even positive slope (snap-back).

• Snap-back is thus expected when $\delta_1 > \delta_2 \Rightarrow B = \frac{s_c}{\tilde{\epsilon}_u l} \leq \frac{1}{3}$. 
The Cohesive Crack Model
FRacture ANALysis Code – ENEL-CRIS Milano and University of Bologna


Catastrophical behaviour
With a CMOD-controlled loading process, it is possible to follow the virtual softening branch BC.

- The energy $W^*$ is released in the time period immediately following the achievement of the peak load.
- It can be measured dynamically through an impulse transducer.

Effect of the structural size-scale

\[ h = 10 \text{cm} \]

\[ h = 20 \text{cm} \]

\[ h = 40 \text{cm} \]

\[ h = 80 \text{cm} \]

• In the nondimensional \( \frac{P}{h} \) plane, the mechanical behaviour is governed by the energy brittleness number, \( s_E = \frac{G^e}{\sigma_u h} \) (Carpinteri, 1985).

\[ \frac{a_0}{h} = 0 \]

\[ \frac{a_0}{h} = 0.5 \]

• Snap-back condition: \( B \leq \frac{1}{3} \) \( \Rightarrow \) \( s_E \leq \frac{E \lambda}{3} = 11.6 \times 10^{-5} \)

\( (\epsilon_{uu} = 0.87 \times 10^{-4}; \lambda = 4) \)
**Size effects on tensile strength**

- The ratio $P_{\text{Cohes}}/P_{\text{US}}$ can be plotted against the nondimensional size, $1/s_E$.
- This ratio represents the ratio of the apparent tensile strength to the true tensile strength (considered as a material constant). It converges to unity for very small values of $s_E$.

![Graph showing the relationship between $P_{\text{Cohes}}/P_{\text{US}}$ and $1/s_E$.]

- The true tensile strength, $\sigma_u$, can be obtained only with very large specimens.

**Comparison between Cohesive Crack Model and design codes**

![Graph comparing the Cohesive Crack Model with Model Code 90 and EC2 (2003).]

- $G_f = 0.1 \text{ kg/cm}$
- $\sigma_u = 20 \text{ kg/cm}^2$
The crucial role of the brittleness number

\[ F_0 = -G \]

low fracture energy \( G_F \)

low tensile strength \( \sigma_u \)

small size \( h \)

large initial crack depth \( a_0/h \)

low slenderness \( l/h \)

\[ s_E = \frac{G_F}{\sigma_u h} \]

low fracture energy \( G_F \)

high tensile strength \( \sigma_u \)

large size \( h \)

small initial crack depth \( a_0/h \)

high slenderness \( l/h \)

The Overlapping Crack Model

Crushing failure

Shear failure

Splitting failure

Quasi-brittle materials show the phenomenon of strain localization also in compression when the elastic limit is overcome


The slenderness- and scale-dependent curves collapse onto a narrow band!

Evaluation of the post-peak overlapping, $w$
In analogy with the Cohesive Crack Model, the Overlapping Crack Model can be defined by a couple of constitutive laws:

We define the **crushing energy** (per unit surface), as the area below the softening curve in the $\sigma - w$ diagram:

$$G_c = 30 \div 60 \text{ N/mm}$$

$$w_c \approx 1 \text{ mm}$$

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**Application to uniaxial compression tests**

Unloaded specimen  
No damage  
Strain localization  
Complete interpenetration

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Experimental assessment: normal strength concrete

\[ \sigma_c = 47.9 \text{ MPa} \]

Numerical
Experimental


Experimental assessment: high strength concrete

\[ \sigma_c = 90.1 \text{ MPa} \]

Numerical Experimental

Structural Ductility

\[ \propto \frac{B}{\varepsilon \lambda} = \frac{G_c}{h \sigma_c \varepsilon \lambda} \]

Constitutive law for reinforcement

Bond-slip relationship (MC90)

The reinforcement reaction and half the crack opening are given, respectively, by the integration of the bond stresses and the relative slips along the transfer length.

**Numerical algorithm for RC beams in bending**

Set consisting in $n$ elastic equations:

$$\{F\} = [K_w]\{w\} + [K_M]M$$

- $\{F\}$ Nodal force vector
- $[K_w]$ Influence coefficients matrix related to the nodal displacements ($w_i = 1$)
- $\{w\}$ Opening/overlapping nodal displacements vector
- $[K_M]$ Influence coefficients vector for the bending moment
- $M$ Applied bending moment

2$n$+1 unknowns: $\{F\}$, $\{w\}$ and $M$

**Additional (n+1) equations:**

- $F_i = 0$ for $i = 1, 2, ..., (j-1)$; $i \neq r$
- $F_i = f(w_i)$ for $i = r$
- $F_i = F_u \left(1 - \frac{w_i}{w_{cr}}\right)$ for $i = j, ..., (m-1)$
- $w_j^r = 0$ for $i = m, ..., p$
- $F_i = F_c \left(1 - \frac{w_i^c}{w_{cr}^c}\right)$ for $i = (p+1), ..., n$

$$F_m = F_u \quad \text{or} \quad F_p = F_c$$
Governing parameter of the process

\[ \text{Position of the fictitious crack tip or } \text{the fictitious overlapping zone tip} \]

\[ \text{Computation of the rotation} \]

At each step:

\[ \vartheta = \{D_w\}^T \{w\} + D_M M \]

\( \{D_w\} \) vector of the coefficients of influence (nodal displacements)

\( \{w\} \) vector of the nodal displacements (opening or overlapping)

\( D_M \) vector of the coefficients of influence (applied moment)

\( M \) applied moment

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Experimental assessment of the proposed model

**Bosco and Debernardi (1993)**

**GEOMETRY**

- \( h = 200, 400, 600 \text{ mm} \)
- \( L/h = 10 \)
- \( \rho_t = 0.13\% - 1.71\% \)
- \( \rho_c = 0.12\% - 0.50\% \)

**CONCRETE**

- \( \sigma_u = 3 \text{ MPa} \)
- \( \sigma_c = 30 \text{ MPa} \)
- \( G_F = 0.065 \text{ N/mm} \)
- \( G_C = 30 \text{ N/mm} \)

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Bending moment-rotation diagrams as functions of the beam size and of the reinforcement percentage

\[ h = 200 \text{ mm} \]


\[ h = 400 \text{ mm} \]
Size-scale effects on the rotational capacity

Numerical results vs. experimental results, $\rho = 1.13\%$

(Bosco and Debernardi 1993)
Effect of steel in compression

Moment vs. rotation response
Horizontal nodal displacements at the maximum rotation

Effect of concrete compressive strength

Moment vs. rotation response
Horizontal nodal displacements at the maximum rotation
Effect of stirrups confinement

Moment vs. rotation response

Horizontal nodal displacements at the maximum rotation

Definition of plastic rotation
Comparison with the prescriptions of Eurocode 2

Eurocode 2: high ductility steel; concrete compressive strength ≤ C50/60.


Size-scale effects on the minimum flexural reinforcement

Numerical vs. experimental results

\[ b = 0.15 \text{ m} \]
\[ h = 0.20 \text{ m} \]
\[ L = 1.20 \text{ m} \]

\[ b = 0.15 \text{ m} \]
\[ h = 0.40 \text{ m} \]
\[ L = 2.40 \text{ m} \]
Definition of minimum reinforcement

The global response is governed by two nondimensional numbers:

\[ N_P = \rho \frac{\sigma_y h^{0.5}}{K_{IC}} \]

\[ S = \frac{K_{IC}}{\sigma_u h^{0.5}} \]

being: \( K_{IC} = \sqrt{G_y E_c} \)

In particular: \( M_{cr} \propto \frac{1}{S} \) and \( M_u \propto N_P \)

Carpinteri A., Cadamuro E., Corrado M. Dimensional analysis approach to the assessment of the minimum flexural reinforcement in RC beams. To appear.
Comparison between different models

Comparison with Design Code formulae
Conditions for structural design
with ductile response

Crushing prevails over steel yielding

Ductile behaviour

Unstable crack propagation

MOMENT

ROTATION